

# Landau-Ginzburg A 模型研究进展\*

范辉军<sup>1</sup>, 蒋文峰<sup>2</sup>, YANG Dingyu<sup>3</sup>

- (1. 北京大学数学科学学院, 北京 100871;
2. 中山大学数学学院 (珠海), 广东 珠海 519082;
3. Humboldt-Universität zu Berlin, Berlin 12489, Germany)

**摘要:** 简要介绍同调镜像对称中的 Landau-Ginzburg A-模型 (LG-A 模型)。首先简要回顾了同调镜像对称的整体图景, 然后讨论了 Landau-Ginzburg 模型的背景以及在同调镜像对称中的应用, 最后, 简要介绍 LG 模型的 Fukaya 范畴的最新研究。本文尝试尽可能地包含数学与物理两方面的背景材料。

**关键词:** Fukaya 范畴; Landau-Ginzburg 模型

**中图分类号:** O186.1    **文献标志码:** A    **文章编号:** 0529-6579 (2020) 01-0001-08

## Progress of the study on Landau-Ginzburg A-model

FAN Huijun<sup>1</sup>, JIANG Wenfeng<sup>2</sup>, YANG Dingyu<sup>3</sup>

- (1. School of Mathematical Sciences, Peking University, Beijing 100871, China;
2. School of Mathematics (Zhuhai), Sun Yat-sen University, Zhuhai 519082, China;
3. Humboldt-Universität zu Berlin, Berlin 12489, Germany)

**Abstract:** A brief introduction of Landau-Ginzburg A-model (LG A-model) in homological mirror symmetry is given. Firstly, a short review of the general picture of the homological mirror symmetry is given. Then the background of Landau-Ginzburg model and its role in homological mirror symmetry are discussed. Finally, a brief introduction of our recent work on the Fukaya category of LG model is included. Both mathematical and physical backgrounds are tried to include in this introduction as much as possible.

**Key words:** Fukaya category; Landau-Ginzburg model

In general, mirror symmetry describes a phenomenon, that the structures of two entirely different mathematical objects are equivalent (in a certain sense). In principle, these objects are two realizations of the same physical theory (often in string theory). In 1990s, Candelas, de la Ossa, Green and Parkes [1] found that we can count the number of algebraic curves via transformations of Picard-Fuchs equation, for quintics in  $\mathbf{P}^4$ . This conjectural result was proved by Givental [2-4] and Liu-Lian-Yau [5-8] respectively in a mathematically rigorous way, and by Hori-Vafa [9] using arguments in physics.

\* 收稿日期: 2019-09-06

**基金项目:** 国家自然科学基金重大项目 (11890661); 国家自然科学基金重点项目 (11831017); 国家自然科学基金 (8200904630); 高校基本科研业务费 (74120-31610002)

**作者简介:** 范辉军 (1972 年生), 男; **研究方向:** 辛几何和数学物理; E-mail: fanhj@math.pku.edu.cn

**通信作者:** 蒋文峰 (1984 年生), 男; **研究方向:** 辛几何与数学物理、图论与概率; E-mail: wen\_feng1912@outlook.com

范辉军, 教授, 博士生导师, 国家自然科学基金杰出青年基金获得者, 教育部长江奖励计划特聘教授, 新世纪百千万人才工程国家级人选, “万人计划”科技创新领军人才, 2017 年获国家自然科学基金二等奖。范辉军与 Jarvis 及阮勇斌合作建立的量子奇点理论目前被国际同行称为 FJRW (范 - Jarvis - 阮 - Witten) 理论。

The aforementioned curve-counting version of mirror symmetry is about the closed string. Here the terminology ‘closed string’ means we consider one dimensional string without boundary, moving in certain dimensional physical world. In Hori-Vafa’s theory, the T-duality plays an important role, which relates strings of scales  $R$  and  $1/R$  and gives a correspondence of invariants on them.

If we consider the string with boundary, there also should be some kind of mirror symmetry phenomenon. It is Kontsevich who proposed a conjecture on open string version in 1994. This program is nowadays known as homological mirror symmetry. Instead of numerical curve counting, the homological mirror symmetry is at the categorical level. For a pair of manifolds mirror to each other, the derived Fukaya category (A-model) of one should be equivalent to the the derived category of coherent sheaves (B-model) of the other, and vice versa.

Here we need some explanation of Fukaya categories, which are constructed from the Lagrangian Floer theory on symplectic manifolds. In general, the objects of the Fukaya category of a symplectic manifold  $M$  are certain Lagrangian submanifolds. For a pair  $(L_0, L_1)$  of such objects, the morphism space  $\text{Hom}(L_0, L_1)$  between them are the module freely generated by their intersections, after intersections of Lagrangians are made transverse. We often use  $CF(L_0, L_1)$  to denote this Hom space.

For  $p_i \in L_{i-1} \cap L_i, i = 1, 2, \dots, d$ , and  $p_0 \in L_0 \cap L_d$ , we consider maps  $u$  from the unit disc to  $M$ , satisfying the Cauchy-Riemann equation

$$\bar{\partial}u := du + J \circ du \circ j = 0$$

The disc here has  $d + 1$  marked points  $z_0, \dots, z_d$  on the boundary, which divide the boundary circle into  $d + 1$  segments. When  $z$  approaches some marked point  $z_i$ , the limit of  $u(z)$  is required to be  $p_i$ . Each boundary segments are required to be mapped to one of Lagrangians, and the order of the Lagrangians is suitably chosen to make the conditions compatible. If certain transversality condition is satisfied, or in general virtual technique is applied to achieve the same affect, the moduli space of such maps consists of component of smooth manifolds. Counting the zero dimensional components gives a number, denoted by  $\langle p_0, p_1, \dots, p_d \rangle$ . We then get the composition map

$$\mu^d : \text{Hom}(L_0, L_1) \times \dots \times \text{Hom}(L_{d-1}, L_d) \rightarrow \text{Hom}(L_0, L_d)$$

which is defined by

$$\mu^d(p_1, \dots, p_d) = \sum_{p \in L_0 \cap L_d} \langle p, p_1, \dots, p_d \rangle p$$

on generators (here we assume no disk or sphere bubbling for simplicity in this review).

The collection of composition maps gives rise to differential graded structure of morphism spaces and their associative compositions up to higher homotopy. In algebraic language, Fukaya category is an  $A_\infty$  category. That is to say, the composition maps will satisfy some  $A_\infty$  relations, which can be verified by the compactification of the moduli spaces. The standard algebraic process of twisting and deriving is then applied to it, in order to access more structures (admitting more objects and making use of higher dimensional moduli spaces).

Mathematically, the best-understood part of homological mirror symmetry is the Calabi-Yau case. In fact, Kontsevich’s conjecture is for mirror symmetry between Calabi-Yau manifolds. There are rich results on different kinds of Calabi-Yau manifolds. For example, elliptic curve [10], Abelian varieties [11], SYZ fibrations [12 – 15], Quartic surfaces [16], Products [17], and Calabi-Yau projective hypersurfaces [18 – 19].

## 1 The Landau-Ginzburg Model

Mirror symmetry is not confined to the scope of Calabi-Yau manifolds. Further examples includes Fano cases. In physicists’ point of view, especially in Hori-Vafa’s notion, mirror symmetry can be interpreted as the correspondence between the ‘gauged linear sigma model’ and the ‘Landau-Ginzburg model’. In general, Landau-Ginzburg model (LG model) introduces a holomorphic function on a Kähler manifold, to study the geometrical and physical

properties of the system. The LG model and the Calabi-Yau version theory are related by Landau-Ginzburg/Calabi-Yau correspondence (LG/CY correspondence). Note that in the closed string version, there is also LG/CY correspondence. The invariants of FJRW theory (constructed in [20 – 22]) of a certain Witten equation are related to Gromov-Witten invariants of a related object.

The global picture of homological mirror symmetry is a square-shaped diagram:

$$\begin{array}{ccc}
 \text{LG A-model} & \leftrightarrow & \text{LG B-model} \\
 \updownarrow & & \updownarrow \\
 \text{CY A-model} & \leftrightarrow & \text{CY B-model}
 \end{array}$$

All two-sided arrows here are conjectured to be equivalences. However, this picture is in principle, that in some cases the LG model might not have a CY corresponding part, and the LG-LG mirror symmetry holds for larger class of examples.

In general, the A-models are string theories, which is mathematically about the Lagrangian Floer theory, which are symplectic geometric studies. The B-models are fields theories, which are depicted in bundles and finally sheaves in abstract version. In [23 – 24], Orlov constructed his open string LG B-model theory, he also established the open string B-model LG/CY correspondence in [25].

In principle, a LG model  $(M, h, W)$  consists of a Kähler manifold  $M$  with metric  $h$ , and a holomorphic function  $W$  on  $M$ . The physical theory in  $(2, 2)$  supersymmetry often uses a quasi-homogenous polynomial  $W$ , to keep some symmetry of the system, but this can also be generalized. The LG A-model on  $M$  is algebraically an  $A_\infty$  category, arising from some Floer-type equation, encoding the information of  $W$ .

The notion of Lefschetz thimble is crucial in our later discussion about this Floer-type equation. So we first elaborate on its definition. For a critical value  $w_0 \in \mathbf{C}$  of  $W$ , we can consider paths  $\gamma: [0, 1] \rightarrow \mathbf{C}$  ending in  $w_0$ , with no intersection with other critical values. These paths are called vanishing paths (associated to  $w_0$ ). If  $W$  is holomorphic Morse, we can regard it as a fibration  $M \rightarrow \mathbf{C}$ . Then for a vanishing path  $\gamma$ , we can naturally define a parallel transport  $\rho_\gamma$ . For  $x \in W^{-1}(w_0)$ , the Lefschetz thimble  $B = B_x$  is defined by

$$B = \{y \in W^{-1}(\gamma(s)), s \in [0, 1] \mid \lim_{t \rightarrow 1} \rho_\gamma|_{[s, t]}(y) = x\} \cup \{x\}$$

This Lefschetz thimble is a Lagrangian submanifold in  $M$ . Given a regular fiber  $N$ , and a path going through its base point, the intersection of a Lefschetz thimble with  $N$  is called a vanishing cycle. Vanishing cycles are  $n - 1$  dimensional, topologically spheres. If we take the vanishing path to be some straight ray parallel to the  $x$ -axis in  $\mathbf{C}$  (instead of segments, we can consider  $\gamma: (-\infty, 0]$  (or  $[0, +\infty)$ )  $\rightarrow \mathbf{C}$  and all above construction can be applied), then the Lefschetz thimble  $B$  is just the stable/unstable manifold of the flow, generated by the vector field  $\nabla \text{Re } W$ . Note that, the imaginary part of  $W$  is constant on  $B$ .

This setting originates from the LG theory in  $(2, 2)$  supersymmetry in physics, where the Lefschetz thimbles play the role of the boundary condition for string worldsheet (see Fig. 1).

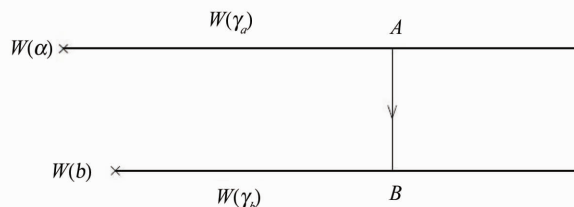


Fig. 1 Strings between two Lefschetz thimbles  $\gamma_a, \gamma_b$ , whose images under  $W$  are parallel rays, where  $a$  and  $b$  are two critical points. This picture is from [26]

One can see [26] for more details. In short, in this setting, we consider paths  $[0, \pi] \rightarrow M$  whose ends sit in two Lefschetz thimbles, respectively. In order to consider the ‘supersymmetric ground states’ of the system, a functional  $h$  is introduced, where we rewrite it as  $\alpha$  for our compatibility of symbols:

$$\alpha(l) = \int_{[0,1] \times [0,\pi]} \hat{l}^* \omega + \int_{[0,\pi]} \operatorname{Re} W(l) ds \quad (1)$$

Here  $\hat{l}$  is a homotopy  $[0,1] \times [0,\pi] \rightarrow M$  that  $\hat{l}(1, \cdot) = l$  and  $\hat{l}(0, \cdot)$  is a fixed path. Note that, we need some condition to make it well defined.

The gradient flow of  $\alpha$  is

$$\partial_s \varphi + J \partial_t \varphi = \nabla \operatorname{Re} W(\varphi) \quad (2)$$

In complex version, it is

$$\partial_z u^j = \frac{1}{2} h^{\bar{j}i} \partial_i \bar{W}$$

where  $h$  is the Kähler metric of  $M$ . This is exactly the Witten equation, which is similar to ones in FJRW theory in [20–22]. However, here we must consider equations with boundary. We will discuss it in the next section.

It is worth noting that, the construction of Lefschetz thimbles and vanishing cycles is not only applied to holomorphic Morse functions  $W$ , but can be used to holomorphic Morse fibrations  $\pi: E \rightarrow S$  for some Riemannian surface  $S$ , and we can still get Lefschetz thimbles and Vanishing circles. In Seidel’s construction of Fukaya-Seidel category (see Fig. 2), one considers  $S$  to be a unit disc, and connects all critical values of  $\pi$  to a fixed point  $z_0$  on the boundary of  $S$ . Then one takes vanishing circles in  $\pi^{-1}(z_0)$  to be the collection of objects, and gets a Fukaya category in  $\pi^{-1}(z_0)$ . The vanishing paths are not canonically chosen, but one can prove some equivalence between different choices. This kind of equivalence is called mutation. One can see [27–29] for reference.

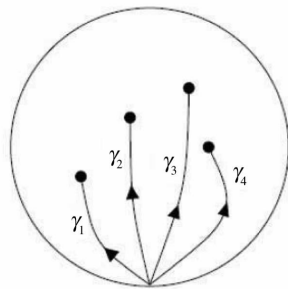


Fig. 2 The setting of Fukaya-Seidel category, in [29]

## 2 Fukaya category of LG model

We need some definitions here, before further discussions. The tuple  $(M, h, W)$  is called a Landau-Ginzburg (LG) system, where  $(M, h)$  is an  $n$ -complex dimensional complete noncompact Kähler manifold and  $W$  is a non-trivial holomorphic function on  $M$ . We also require that  $M$  is of bounded geometry, using the Kähler metric. If the Kähler form  $\omega$  of the metric  $h$  is exact, we call  $(M, h, W)$  an exact LG system.

Given a LG system  $(M, h, W)$ , we say that it satisfies the tame condition if there exist a base point  $q_0 \in M$  and constants  $C_1, C_2, \delta > 0$  such that for any point  $\varphi = (u_1, \dots, u_n, \bar{u}_1, \dots, \bar{u}_n) \in M$

$$\begin{cases} \text{(i)} & |W(u)| + |\nabla_M^2 W(u)| \leq C_1 d(\varphi, q_0) + |\partial_M W(u)| + C_2 \\ \text{(ii)} & d(\varphi, q_0) \leq C_1 |\partial_M W(u)| + C_2 \\ \text{(iii)} & |\partial_M W(u)| \leq C_1 e^{\delta d(\varphi, q_0)} + C_2 \end{cases} \quad (3)$$

Another concept we introduce is a regular LG system. If  $W$  is Morse, and for any pair  $p, q \in C_W, p \neq q$ , holds  $\text{Im } W(p) \neq \text{Im } W(q)$ , then the LG system  $(M, h, W)$  is called a regular Morse LG system. A regular tame exact LG system  $(M, h, W)$  is a regular LG system, which has exact Kähler form  $\omega$  and satisfies the tame condition.

Note that the condition of exactness will make the functional in (1) well defined.

We will define the LG-Fukaya category of such a regular tame exact LG system. The objects of the category are Lefschetz thimbles with extra data on them. In this context, Lefschetz thimbles are stable manifolds of the flow generated by the gradient of  $\text{Re } W$ . Note that, the conditions above will make these Lefschetz thimbles finite in number and disjoint. We lift the Lefschetz thimbles to the universal Abelian cover  $\widetilde{GM}$  of the bundle  $GM$  of the Lagrangian subspaces of  $TM$ .

A Landau-Ginzburg Lagrangian brane  $L^\#$  is a Lefschetz thimble  $L$ , together with a lift  $i^\#: L \rightarrow L^\#$  and a Pin structure on  $TL$ . These Landau-Ginzburg Lagrangian branes are the objects of our category.

To define the morphism between objects  $L_0^\#, L_1^\#$ , we consider the equation

$$\frac{\partial l}{\partial t} = X_W \tag{4}$$

whose local coordinate form is

$$i \cdot \frac{\partial z^l(t)}{\partial t} = \sum_i h^{i\bar{i}} \partial_i \bar{W}$$

The solution maps of ‘open strings’  $[0, T] \rightarrow M$  for such an equation with  $l(0) \in L_0, l(T) \in L_1$  are called Hamiltonian chords between  $L_0$  and  $L_1$ , and they freely generate the Hom space  $\text{Hom}(L_0^\#, L_1^\#)$ . This space is graded, by topological information of  $L_0^\#, L_1^\#$  (not only  $L_0, L_1$ ), and it is a  $H_1(GM)$  graded module.

In fact, in order to define our  $A_\infty$  category, we need to consider strings with different ‘speeds’. That is to say, instead of (4), we need to consider

$$\frac{\partial l}{\partial t} = \kappa X_W \tag{5}$$

with  $\kappa = 1, 2, \dots$ . The set of solutions with a fixed  $\kappa$  is denoted by  $S_{\kappa, W}(L_0, L_1)$ , and  $S_W(L_0, L_1) = \cup_\kappa S_{\kappa, W}(L_0, L_1)$ . First, we need these solutions to be finite in number for each  $\kappa$ , to get a well defined theory. If so, we say that,  $L_0$  and  $L_1$  intersect transversely at time  $T$  with speed  $\kappa$ . We can choose generic Kähler metric to get a transverse intersection result, the metric should lie in some space, denoted by  $\mathcal{S}_\mu^L$  here.

**Theorem 1** (Theorem 3. 12 of [30]) Let  $(M, h, W)$  be a tame LG system. For a residual metric in  $\mathcal{S}_\mu^L$  and a residual  $T \in (1/2, 2)$ ,  $L_0$  and  $L_1$  intersect transversely at time  $T$  with speed  $\kappa$  for each  $\kappa$ .

We choose a residual  $T$ , and denote  $[0, T]$  by  $[0, 1]$  by notational convention. In this setting the Hom space  $\text{Hom}(L_0^\#, L_1^\#)$  is generated by  $S_W(L_0, L_1)$ , the grading for this space is enlarged by  $\mathbf{Z}$ , one can see [30, 8. 2] for more detail for the grading. We denote the Hom space by  $CF(L_0^\#, L_1^\#)$ .

To define composition maps, we must define at first the Witten equation for the pointed discs. Techniqually, to define the equation, we first need a section  $\sigma$  of the log-canonical bundle of the disc. First,  $\sigma$  needs to be holomorphic. In addition, the imaginary part  $\text{Im}\sigma$  should be zero on the boundary. Note that, on strip-like ends, where the part of the disc can be written as  $[a, +\infty) \times [0, 1]$  (or  $(-\infty, -a] \times [0, 1]$ ) for some  $a > 0$ ,  $\sigma$  can be written as  $\kappa dz$  for some  $\kappa \in \mathbf{R}^+$  in this coordinate.

Index formula determines that, the space of such sections is one dimensional. For branes  $L_0^\#, L_1^\#, \dots, L_d^\#$ , we consider the maps  $\phi$  from  $d + 1$  pointed disc to  $M$ , and the equation:

$$\bar{\partial}_j \phi = (X_{\text{Re } W} \text{Im}\sigma + Z(\phi))^{0,1} \tag{6}$$

Here,  $Z$  is the perturbation term. Marked points  $z_0, \dots, z_d$  on the boundary divide the boundary circle into  $d + 1$  seg-

ments. For  $l_i \in S_W(L_{i-1}, L_i), i = 1, \dots, d$  and  $l_0 \in S_W(L_0, L_d)$ , we can consider such solutions  $\phi$  that the limit of  $\phi(z)$  is  $l_i$  when  $z \rightarrow z_i$ . We also require that the boundary segments are mapped to one of these Lefschetz thimbles, whose order is suitably chosen to make the conditions compatible. Note that, when  $d = 1$ , we can regard the pointed disc as a strip  $\mathbf{R} \times [0, 1]$ , and the equation (6) becomes

$$\partial_s \phi + J \partial_t \phi = \kappa \nabla \operatorname{Re} W + z(\phi)$$

which is similar to (2), except a different coefficient  $\kappa$  and a perturbation  $z(\phi) = Z(\phi) \left( \frac{\partial}{\partial s} \right)$ . This implies that the equation can be regarded as some kind of gradient flow.

We take the change of the configuration of discs into consideration and find that, for generic perturbations, such maps will make up a smooth manifold of a certain dimension. If the dimension is zero, it is called the rigid one, then counting such maps will give a number  $\langle l_0, l_1, \dots, l_d \rangle$ . Counting these numbers will give rise to composition maps

$$\mu^d : \operatorname{Hom}(L_0^\#, L_1^\#) \times \dots \times \operatorname{Hom}(L_{d-1}^\#, L_d^\#) \rightarrow \operatorname{Hom}(L_0^\#, L_d^\#)$$

where on generators, we have

$$\mu^d(l_1, \dots, l_d) = \sum_{l \in S_W(L^0, L^d)} (-1)^* \langle l, l_1, \dots, l_d \rangle l$$

The symbol  $*$  is used to give a sign of the component. By studying the compactness of the moduli spaces, we get the  $A_\infty$  relation (132 of [30])

$$\sum_{i+j+k=d} (-1)^\dagger \mu^{i+k+1}(l_1, \dots, l_i, \mu^j(l_{i+1}, \dots, l_{i+j}), l_{i+j+1}, \dots, l_{i+j+k}) = 0 \quad (7)$$

where  $(-1)^\dagger \mu$  is an integer according to these inputs, which is determined by the coherent orientations of the moduli spaces.

However, the compactness is far from a trivial result. This is because, the target  $M$  is not compact. So we need some kind of  $C^0$  estimate. This is done by a mutual control mechanism between the  $C^0$  bound of  $\phi$  and  $|d\phi|$ , for which one can see [30] for the detail. In short, we need to examine the ‘bubbling’ phenomenon where  $|d\phi| \rightarrow +\infty$ . However, as it is noncompact, we can not get an actual bubble, for no convergence can be obtained if we carry out the bubbling process. Instead, we use some kind of elliptic estimate to control  $|d\phi|$ , under the condition that the total energy is bounded. In this process, isoperimetric inequality is used (together with the exactness of the symplectic form associated to  $h$  and bounded geometry of  $(M, h)$ ), and the tame condition is also crucial to control the gradient term  $X_{\operatorname{Re} W} \operatorname{Im} \sigma + Z(\phi)$ .

Summing up, we get the Fukaya category  $\operatorname{Fuk}(M, h, W)$  of a Landau-Ginzburg model  $(M, h, W)$ , which consists of the following data:

- (i) A set  $\operatorname{Ob}(\operatorname{Fuk}(M, h, W))$  of objects, consisting of all Landau-Ginzburg branes.
- (ii) For each pair  $(L_0^\#, L_1^\#)$  of Landau-Ginzburg branes, a morphism space

$$\operatorname{Hom}(L_0^\#, L_1^\#) = CF(L_0^\#, L_1^\#)$$

- (iii) Composition maps

$$\mu^d : \operatorname{Hom}(L_0^\#, L_1^\#) \otimes \dots \otimes \operatorname{Hom}(L_{d-1}^\#, L_d^\#) \rightarrow \operatorname{Hom}(L_0^\#, L_d^\#)$$

satisfying the  $A_\infty$  relation (7).

### 3 More discussion

There are many motivations of the construction of this Fukaya category of LG model. On the other hand, we can also expect to have many applications of this theory. We present a brief discussion in this section.

The readily goal of this construction is to extend the homological mirror symmetry to more general cases. There

are already many studies of homological mirror symmetry of LG model, one can see the introduction in [30] for a comprehensive summary. Our Fukaya category of LG model is defined on a general class of Kähler manifolds (exact, with bounded geometry). Once it is defined, we can expect that, it can be used to extend the homological mirror symmetry to more general cases (beyond Fano case).

It worth noting that, if we choose a regular fiber  $\pi^{-1}(w_0)$ , we use Seidel's construction to get a Fukaya-Seidel category  $\text{Fuk}(\pi^{-1}(w_0))$ . A natural question is the relation between  $\text{Fuk}(\pi^{-1}(w_0))$  and  $\text{Fuk}(M, h, W)$ , as they are both A-side theory of LG model. There is no evidence for general relation here. However, in special cases, as  $\mathbf{C}^n$  and  $(\mathbf{C}^*)^n$ , we expect there is some  $A_\infty$  quasi-equivalence between them. Some construction is done, in a work in progress by H. Fan, W. Jiang and D. Yang. Furthermore, the Fukaya category of LG model constructed here can be expected to 'unify' different constructions for LG A-model ever appeared.

This theory is also related to the quantum singularity theory via Witten equation developed in [20–22] (commonly referred to as FJRW theory), which is a closed string invariant about singularity, constructed by studying the Witten equation on orbifold line bundles on closed Riemannian surfaces. Actually, the construction of LG Fukaya category draws many inspirations from the FJRW theory. In particular, the tame condition we used in the construction above is from there. To go further, we can expect to construct an enriched theory to take both boundary marked points in LG Fukaya category theory, and interior marked points in FJRW theory into consideration. This 'universal' theory, once established, can be used to formulate some kind of open-closed correspondence of LG model.

In addition, in Gaiotto-Moore-Witten's web-based formalism ([31]) of LG theory for holomorphic Morse  $W$ , one can consider polytopes generated by singular values of  $W$  in  $\mathbf{C}$  and secondary fans of all its possible regular polyhedral subdivisions (the dual of which is the space of webs), and establish an  $L_\infty$ -algebra  $\mathcal{B}$ . One can find the mathematical formulation in the paper of Kapranov-Kontsevich-Soibelman ([32]). The construction there is expected to recover some kind of LG-Fukaya category ([32, Conjecture 14.10]). In this paper, we are using the same Witten equation as in GMW and KKS, and techniques and viewpoints especially compactness etc. should pave the way to rigorously construct this algebra of infrared and approach this conjecture.

## Reference:

- [1] CANDELAS P, OSSA X C D L, GREEN P S, et al. A pair of Calabi-Yau manifolds as an exactly soluble superconformal theory [J]. Nuclear Physics B, Particle Physics, 1991, 359(1): 21–74.
- [2] GIVENTAL A. A mirror theorem for toric complete intersections [J]. Topological Field Theory, Primitive Forms and Related Topics, 1998: 141–175.
- [3] GIVENTAL A. Elliptic Gromov-Witten invariants and the generalized mirror conjecture [J]. Integrable systems and algebraic geometry (Kobe/Kyoto, 1997), 1997: 107–155.
- [4] GIVENTAL A. Homological geometry and mirror symmetry [C]//Proceedings of ICM, 1994.
- [5] LIAN B, LIU K, YAU S. Mirror principle, I [J]. Asian Journal of Mathematics, 1997, 1(4): 729–763.
- [6] LIAN B, LIU K, YAU S. Mirror principle, II [J]. Asian Journal of Mathematics, 1999, 3(a): 109–146.
- [7] LIAN B, LIU K, YAU S. Mirror principle, III [J]. Asian Journal of Mathematics, 1999, 3(b): 771–800.
- [8] LIAN B, LIU K, YAU S. Mirror principle, IV [J]. Surveys in Differential Geometry, 2000, 7: 475–496.
- [9] HORI K, VAFA C. Mirror symmetry [J]. ArXiv, 2000. [arXiv:hep-th/0002222].
- [10] POLISHCHUK A, ZASLOW E. Categorical mirror symmetry in the elliptic curve [J]. Winter School on Mirror Symmetry, Vector Bundles and Lagrangian Submanifolds (Cambridge, MA, 1999), 1999: 275–295.
- [11] FUKAYA K. Mirror symmetry of abelian varieties and multi-theta functions [J]. J Algebraic Geom, 2002, 11(3): 393–512.
- [12] KONTSEVICH M, SOIBELMAN Y. Homological mirror symmetry and torus fibrations [J]. Symplectic Geometry and Mirror Symmetry (Seoul, 2000), 2000: 203–263.

- [13] FUKAYA K. Floer homology for families – a progress report [J]. *Integrable Systems, Topology, and Physics (Tokyo, 2000)*, 2000: 33 – 68.
- [14] ABOUZAID M. Family Floer cohomology and mirror symmetry [J]. *Proceedings of the International Congress of Mathematicians – Seoul 2014*, 2014, II: 813 – 836.
- [15] TU J. On the reconstruction problem in mirror symmetry [J]. *Adv Math*, 2014, 256: 449 – 478.
- [16] SEIDEL P. Homological mirror symmetry for the quartic surface [J]. *Mem Amer Math Soc*, 2003, 236 (1116): 1 – 129.
- [17] ABOUZAID M, SMITH I. Homological mirror symmetry for the 4 – torus [J]. *Duke Math J*, 2010, 152(3): 373 – 440.
- [18] SEIDEL P. Fukaya categories and deformations [C]//*Proceedings of the International Congress of Mathematicians, Vol II (Beijing, 2002)*, 2002: 351 – 360.
- [19] SHERIDAN N. Homological mirror symmetry for Calabi-Yau hypersurfaces in projective space [J]. *Invent Math*, 2015, 199 (1): 1 – 186.
- [20] FAN H, JARVIS T, RUAN Y. Geometry and analysis of spin equations [J]. *Comm Pure Applied Math*, 2008, 61: 715 – 788.
- [21] FAN H, JARVIS T, RUAN Y. The Witten equation, mirror symmetry and quantum singularity theory [J]. *Ann of Math*, 2013, 178(1): 1 – 106.
- [22] FAN H, JARVIS T, RUAN Y. The Witten equation and its virtual fundamental cycle [J]. *Mathematics*, 2011.
- [23] ORLOV D. Triangulated categories of singularities and D-branes in Landau-Ginzburg models [J]. *Proc Steklov Inst Math*, 2004, 246(3): 227 – 248.
- [24] ORLOV D. Matrix factorizations for nonaffine LG-models [J]. *Math Ann*, 2012, 353(1): 95 – 108.
- [25] ORLOV D. Derived categories of coherent sheaves and triangulated categories of singularities [J]. *Algebra, Arithmetic, and Geometry*, 2010, 270: 503 – 531.
- [26] HORI K, IQBAL A, VAFA C. D-branes and mirror symmetry [J]. *ArXiv*, 2000. [arXiv:hep – th/0005247].
- [27] SEIDEL P. Vanishing cycles and mutation [M]. *European Congress of Mathematics*, Birkhäuser Basel, 2001: 65 – 85.
- [28] SEIDEL P. More about vanishing cycles and mutation [J]. *Symplectic Geometry and Mirror Symmetry (Seoul, 2000)*, 2000: 429 – 465.
- [29] SEIDEL P. *Fukaya categories and Picard-Lefschetz theory* [M]. *European Mathematical Society*, 2008.
- [30] FAN H, JIANG W, YANG D. Fukaya category of Landau-Ginzburg model [J]. *ArXiv*, 2018. [arXiv:1812.11748].
- [31] GAIOTTO D, MOORE G, WITTEN E. Algebra of the infrared: string field theoretic structures in massive  $N = (2, 2)$  field theory in two dimensions [J]. *ArXiv* 2015. [arXiv:1506.04087].
- [32] KAPRANOV M, KONTSEVICH M, SOIBELMAN Y. Algebra of the infrared and secondary polytopes [J]. *Advances in Mathematics*, 2016, 300: 616 – 671.

(责任编辑 冯兆永)